Quantum aspects in gravitational collapse: non-locality and non-singularity

Dejan Stojkovic
HEPCOS, Department of Physics, SUNY at Buffalo, Buffalo, NY 14260-1500, USA

We study the end stages of gravitational collapse of the thin shell of matter in ingoing Eddington-Finkelstein coordinates. We use the functional Schrodinger formalism to capture quantum effects in the near singularity limit. We find that that the equations of motion which govern the behavior of the collapsing shell near the classical singularity become strongly non-local. This reinforces previous arguments that quantum gravity in the strong field regime might be non-local. We managed to solve the non-local equation of motion for the dust shell case, and found an explicit form of the wavefunction describing the collapsing shell. This wavefunction and the corresponding probability density are non-singular at the origin, thus indicating that quantization should be able to rid gravity of singularities, just as it was the case with the singular Coulomb potential.

I. INTRODUCTION

What happens at the last stages of the gravitational collapse of some distribution of matter is still unknown. The reason is our lack of a fully fledged theory of quantum gravity which will fatefully describe quantum dynamics in very strong gravitational fields (e.g. near classical singularities). Since the formulation of quantum gravity still seems to be far from our reach, we have to work with what we have at hand, and try to push it as far as possible. Along the way, we might get a glimpse of what the ultimate theory of quantum gravity should look like.

The purpose of this paper is to study quantum aspects of gravitational collapse of a shell of matter in the context of the functional Schrodinger formalism [1–14]. We will work in Eddington-Finkelstein coordinates which are convenient for studying the question of the black hole formation till the very end where the collapsing matter distribution crosses its own Schwarzschild radius and starts approaching the classical singularity at the center. The first interesting finding is that the equations of motion describing behavior of the collapsing shell near the classical singularity become non-local. It has been argued for a while that (for various reasons) quantum gravity should ultimately be a manifestly non-local theory [8, 9, 15–17], (see also [18–20]). Our finding is a strong indication that something like that might indeed be true. While the functional Schrodinger formalism is not a full theory of quantum gravity, it should however capture some aspects of it. Non-locality might be one of those important aspects.

Non-local equations are notoriously difficult to solve. However, manipulating the equations of motion in the near singularity limit, we managed to find an explicit solution to the non-local Schrodinger equation. Interestingly enough, the solution for the wavefunction is non-singular at the origin. In fact, the probability density becomes zero exactly at the origin. This indicates that quantization can perhaps remove classical singularities from gravity, as argued from many different points of view [8, 9, 21–27].

II. THE SETUP

In this section we will setup the metric of a collapsing shell of matter in ingoing Eddington-Finkelstein coordinates. Since this space-time foliation is non-singular at the Schwarzschild radius, it will allow us to study the the gravitational collapse as the shell is approaching the classical singularity at the center.

The radius of a collapsing spherically symmetric shell of mater is R. The parameter of evolution is the ingoing null coordinate v related to the asymptotic Schwarzschild time as

$$v = t + r^* \tag{1}$$

where r^* is the tortoise coordinate. The trajectory of the collapsing shell is then simply $r = R(v)$. The metric outside the collapsing shell is

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \quad r > R(v). \tag{2}$$

By Birkhoff theorem, the interior metric is Minkowski

$$ds^2 = -dT^2 + dr^2 + r^2\Omega^2, r < R(v) \tag{3}$$

The interior time coordinate, T, is related to the ingoing null coordinate, v, via the proper time on the shell, τ.

The quantity M in Eq. (4) is an integral of motion, and has a clear interpretation of the total energy.

$$M = \mu\left(1 + {R_\tau}^2\right)^{\frac{1}{2}} - \frac{\mu^2}{2R} \tag{4}$$

It contains the rest mass of the shell μ, the kinetic energy represented by R_τ, and gravitational self-energy $\mu^2/(2R)$. We will therefore identify it with the Hamiltonian of the system.[The subscripts here refer to the derivative with the respect to the corresponding coordinate, i.e. $R_\tau = \partial R/\partial\tau$.]

We now express the R_τ in terms of R_T to obtain

$$M = \mu\left(\frac{1}{\sqrt{1 - {R_T}^2}} - \frac{\mu}{2R}\right) \tag{5}$$

The Hamiltonian is now

$$H = \mu(B - R_v)\left(\frac{1}{\sqrt{B - 2R_v}} - \frac{\mu G}{2R}\frac{1}{\sqrt{{R_v}^2 - 2R_v + B}}\right). \tag{6}$$

where $B = 1 - R_s/R$, $R_s = 2GM$, and $R_v = \partial R/\partial v$. We emphasis that so far we did not use any approximations, so the Hamiltonian in Eq. (6) is exact.

III. QUANTUM COLLAPSE OF THE SHELL IN THE LIMIT OF $R \to 0$

The main goal of the this paper is to see what happens at the last stages of the collapse of the shell, i.e. when $R \to 0$. Since we have an explicit Hamiltonian of the system, we can apply the functional Schrodinger formalism and study quantum effects near the classical singularity. In the framework of the functional Schrodinger formalism, we will simply write down the Schrodinger equation for the wave-functional $\Psi[R(v)]$, and try to solve it.

We first derive the behavior of R_τ near $R = 0$. From Eq. (4), we have

$$R_\tau = \sqrt{\left(\frac{M}{\mu} + \frac{\mu G}{2R}\right)^2 - 1}. \tag{7}$$

From here we see that $R_\tau \approx \frac{\mu G}{2R}$ as R is approaching zero. Substituting this result in Eq.(7), we find

$$R_v \approx -\frac{1}{2}\left(\frac{\mu G}{R}\right)^2 \tag{8}$$

Thus, the rate at which the dust shell collapses near $R = 0$ diverges as $R_v \propto \frac{1}{R^2}$.

In this limit the Hamiltonian in Eq. (6) can be approximated as

$$H = \mu(-R_v)\left[\frac{1}{\sqrt{2\mid R_v\mid}} - \frac{\mu G}{2R}\frac{1}{R_v}\right] \tag{9}$$

which gives

$$R_v = 2\left(\frac{H}{\mu} - \frac{G\mu}{2R}\right)^2 \tag{10}$$

For $R \to 0$, we can ignore the constant term H/μ, and we will again get Eq. (8).

In the limit $R \to 0$, the canonical momentum reduces to

$$\Pi = \mu\left[\frac{1}{\sqrt{2\mid R_v\mid}} + \frac{\mu G}{2R}\right] \tag{11}$$

Expressing R_v in terms of Π in Hamiltonian (9) we get

$$H = \frac{-R}{G}\left[1 - \frac{2\Pi R}{\mu^2 G}\right]^{-1} + \frac{\mu^2 G}{2R}. \tag{12}$$

The Hamiltonian in Eq. (12) governs the evolution of the collapsing dust shell in vicinity of $R = 0$. As in the standard quantization procedure, we promote the momentum Π into an operator

$$\Pi = -i\hbar\frac{\partial}{\partial R}. \tag{13}$$

We can now write the functional Schrodinger equation for the wave-functional $\psi[R(v)]$

$$H\psi = i\hbar\frac{\partial\psi}{\partial v} \tag{14}$$

and try to solve it. Unfortunately, the structure of the Hamiltonian (12) is such that the usual treatment is practically impossible. The main problem is that the differential operator in Hamiltonian (12) is non-local. This finding represents a strong support for suggestions that quantum gravity might be ultimately a non-local theory.

While finding solutions to non-local equations is very difficult, we will show that it is possible to define a procedure (similar to the one outlined in [5]) which will lead to the solution of Eq. (14). We first isolate the non-local operator $\hat{A}$ from the Hamiltonian (12)

$$\hat{A} = \left[1 - \frac{2\Pi R}{\mu^2 G}\right]^{-1} \tag{15}$$

Its inverse is

$$\hat{A}^{-1} = 1 - \frac{2\Pi R}{\mu^2 G} \tag{16}$$

We can take care of the operator ordering as

$$\hat{A} = \left[1 - \frac{1}{\mu^2 G}(\hat{\Pi}R + R\hat{\Pi})\right]^{-1}, \tag{17}$$

so that

$$\hat{A}^{-1} = 1 - \frac{1}{\mu^2 G}(\hat{\Pi}R + R\hat{\Pi}). \tag{18}$$

In terms of derivatives, $\hat{A}^{-1}$ is

$$\hat{A}^{-1} = (1 + \frac{i}{\mu^2 G}) + \frac{2iR}{\mu^2 G}\frac{\partial}{\partial R} \tag{19}$$

Let's define the action of an operator $\hat{A}$ as $\varphi = \hat{A}\psi$, which means $\psi = \hat{A}^{-1}\varphi$, where φ is just some function which gives the wavefunction ψ upon action of the operator $\hat{A}$. Explicit action of $\hat{A}^{-1}$ on φ converts the equation $\hat{A}^{-1}\varphi = \psi$ into a linear differential equation

$$\frac{d\varphi}{dR} + \frac{1}{2R}(1 - i\mu^2 G)\varphi + \frac{i\mu^2 G}{2R} = 0 \tag{20}$$

This equation can be solved to give

$$\varphi = -\frac{i\mu^2 G}{2}\frac{\int R^{-\frac{(1+i\mu^2 G)}{2}}\psi dR}{R^{\frac{(1-i\mu^2 G)}{2}}} \tag{21}$$

Since $\varphi = \hat{A}\psi$ we obtain the action of $\hat{A}$ as

$$\hat{A} = -\frac{i\mu^2 G}{2} \frac{\int R^{-\frac{(1+i\mu^2 G)}{2}}(.)dR}{R^{\frac{(1-i\mu^2 G)}{2}}} \tag{22}$$

where (.) is the placeholder for the function on which $\hat{A}$ is acting.

Let's concentrate on the stationary solutions to Eq. (14) in the form of

$$\psi(R,v) = \psi(R)e^{iEv/\hbar} \tag{23}$$

where v is the time evolution parameter, and E is the energy eignevalue. The time independent Schrodinger equation becomes $H\psi = E\psi$. The Hamiltonian in Eq. (12) in terms of the operator $\hat{A}$ becomes

$$H = \frac{R\hat{A}}{G} + \frac{\mu^2 G}{2R} \tag{24}$$

Accounting for the ordering of operators, this Hamiltonian becomes

$$H = -\frac{1}{2G}\left(R\hat{A} + \hat{A}R\right) + \frac{\mu^2 G}{2R}. \tag{25}$$

The Schrodinger equation (14) becomes

$$\frac{-1}{2G}\left(R\hat{A} + \hat{A}R\right) + \frac{\mu^2 G}{2R} = E\psi \tag{26}$$

When $\hat{A}$ operates on R we get

$$\hat{A}(R) = -\frac{\alpha}{2}R^{-\frac{1}{2}(1-\alpha)} \int R^{-\frac{1}{2}(1+\alpha)} R dR \tag{27}$$

which yields

$$\hat{A}(R) = \frac{-\alpha R + \beta}{3 - \alpha} \tag{28}$$

where $\alpha = i\mu^2 G$ and β is an integration constant. So our equation becomes

$$\frac{-1}{2G}\left(R\hat{A}\psi + \frac{\alpha R - \beta}{\alpha - 3}\right) + \frac{\mu^2 G}{2R} = E\psi \tag{29}$$

Now we can move all the terms to one side and separate the term with the integral

$$\int R^{-\frac{1}{2}(1+\alpha)}\psi dR = \frac{4GR^{-\frac{1+\alpha}{2}}}{\alpha}\left[\frac{1}{2G}\left(\frac{\alpha R - \beta}{\alpha - 3}\right) - \frac{\mu^2 G}{2R} + E\right] \tag{30}$$

We can now differentiate this equation with respect to R to remove integration. Differentiation yields

$$\left[\frac{2R^{\frac{1}{2}(1-\alpha)}}{\alpha - 3} - 2\mu^2 G^2 R^{-\frac{1}{2}(3+\alpha)} + \left(\frac{4GE}{\alpha} - \frac{2\beta}{\alpha(\alpha-3)}\right) R^{-\frac{1}{2}(1+\alpha)}\right]' = \tag{31}$$
$$\left[\left(\frac{\alpha - 1}{\alpha - 3} + 1\right) R^{-\frac{1}{2}(1+\alpha)} - \mu^2 G^2(\alpha+3)R^{-\frac{1}{2}(5+\alpha)} + \left(\frac{2GE(\alpha+1)}{\alpha} - \frac{\beta(\alpha+1)}{\alpha(\alpha-3)}\right) R^{-\frac{1}{2}(3+\alpha)}\right]$$

This can be written as

$$\frac{d\psi}{} = \int \frac{a_1 + a_2 R + a_3 R^2}{a_4 R + a_5 R^2 + a_6 R^3} dR \tag{32}$$

where $a_1 = -\mu^2 G^2(\alpha+3)$, $a_2 = \left(\frac{2GE(\alpha+1)}{\alpha} - \frac{\beta(\alpha+1)}{\alpha(\alpha-3)}\right)$, $a_3 = 1 + \frac{\alpha-1}{\alpha+3}$, $a_4 = -2\mu^2 G^2$, $a_5 = \left(\frac{4GE}{\alpha} - \frac{2\beta}{\alpha(\alpha-3)}\right)$ and $a_6 = \frac{2}{\alpha-3}$ This integral can be solved for general values of constants. However, since we are working in the limit of $R \approx 0$, we keep only the leading order terms

$$\ln\psi = \int \frac{a_1}{a_4 R} dR + \text{constant} \tag{33}$$

Solving this equation and substituting the values of the constants, we find the solution for the wavefunction

$$= \lambda R^{\frac{3+i\mu^2 G}{2}}. \tag{34}$$

where λ is a constant. The corresponding probability density $P = \psi^*\psi$ is

$$|\psi|^2 = \lambda^2 R^3. \tag{35}$$

This result is very important. It demonstrates that the probability density associated with the wavefunction ψ which describes the collapse of the shell of matter is non-singular near the classical singularity. In fact, the probability density in Eq. (35) vanishes exactly at $R = 0$. It is remarkable that a simple quantum treatment of the gravitational collapse indicates that classical singularity at the center can be removed.

IV. CONCLUSIONS

In this paper we studied quantum aspects of the gravitational collapse near the classical singularity as seen by an infalling observer. Since gravity is the by far the weakest force in nature, we expect that quantum mechanics will significantly modify classical behavior of gravity only in the strong field regimes, e.g. near classical singularities. In the absence of a fully fledged theory of quantum gravity, we worked in the context of the functional Schrodinger formalism applied to a simple gravitational system - collapsing shell of matter. We used the Eddington-Finkelstein space-time foliation which is convenient for studying the question of the black hole formation till the very end where the collapsing shell crosses its own Schwarzschild radius and starts approaching the classical singularity at the center. We derived the conserved quantity with the clear interpretation as the Hamiltonian of the system and quantized the theory. In the $R \to 0$ limit, we found that the equation which describes the quantum evolution of the collapsing shell is strongly non-local. Non-local terms which are usually suppressed in large distance limit, become dominant in the near singularity limit. This conforms some earlier speculations and related studies. As an important step forward, we managed to solve this non-local equation explicitly and found the form of the wavefuction. Remarkably, the wavefunction and its corresponding probability density are non-singular at $R \to 0$. This is an indication that quantization can remove classical singularities from gravity, just as it was the case with the singular electromagnetic Coulomb potential.

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